Triple Integrals in Cylindrical and Spherical Coordinates

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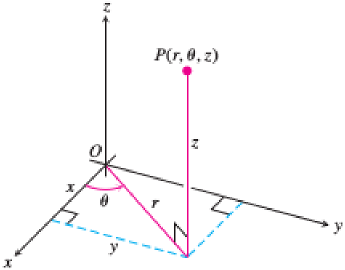
Overview

When a calculation in physics, engineering, or geometry involves a cylinder, cone, sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in the lecture.

The procedure for transforming to these coordinates and evaluating the resulting triple integrals is similar to the transformation to polar coordinates in the plane discussed earlier.

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis.

This assigns to every point in space one or more coordinate triples of the form (r, θ, z).



Definition 1.

Cylindrical coordinates *represent a point P in space by ordered triples (r, θ, z) in which*

1. *r and θ are polar coordinates for the vertical projection of P on the xy-plane*
2. *z is the rectangular vertical coordinate.*

The values of x, y, r, and θ in rectangular and cylindrical coordinates are related by the usual equations.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates :

$$x = r\ \cos\theta, \quad y = r\ \sin\theta, \quad z = z,$$

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x.$$

Constant-coordinate Equations in Cylindrical Coordinates

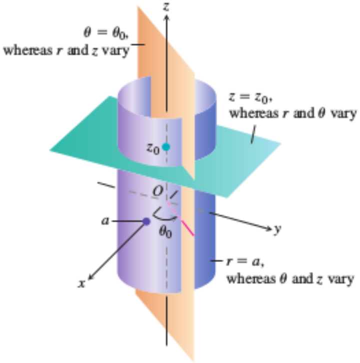
In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis.

The z-axis is given by $r = 0$.

The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

Thus constant-coordinate equations in cylindrical coordinates yield cylinders and planes.



Cylindrical Coordinates

Cylindrical coordinates are good for describing cylinders whose axes run along the z-axis and planes the either contain the z-axis or lie perpendicular to the z-axis.

Surfaces like these have equations of constant coordinate values:

$r = 4$	Cylinder, radius 4, axis the z-axis
$\theta = \pi/3$	Plane containing the z-axis
$z = 2$	Plane perpendicular to the z-axis

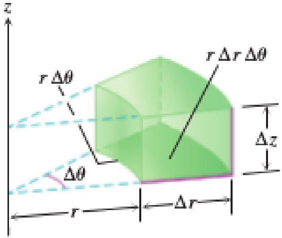
When computing triple integrals over a region D in cylindrical coordinates, we partition the region into n small cylindrical wedges, rather than into rectangular boxes.

Cylindrical Coordinates

In the kth cylindrical wedge, r, θ and z change by $\Delta r_k, \Delta\theta_k$, and Δz_k, and the largest of these numbers among all the cylindrical wedges is called the **norm** of the partition.

We define the triple integral as a limit of Riemann sums using these wedges.

The volume of such a cylindrical wedge ΔV_k is obtained by taking the area ΔA_k of its base in the $r\theta$-plane and multiplying by the height Δz.



Cylindrical Coordinates

For a point (r_k, θ_k, z_k) in the center of the kth wedge, we calculated in polar coordinates that $\Delta A_k = r_k \ \Delta r_k \ \Delta\theta_k$. So $\Delta V_k = \Delta z_k \ r_k \ \Delta r_k \ \Delta\theta_k$ and a Riemann sum for f over D has the form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k) \Delta z_k \ r_k \ \Delta r_k \ \Delta\theta_k.$$

The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero

$$\lim_{n\to\infty} = \iiint\limits_D f \ dV = \iiint\limits_D f \ dz \ r \ dr \ d\theta.$$

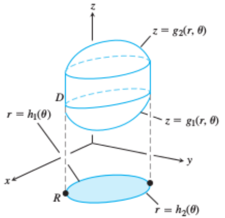
Triple integrals in cylindrical coordinates are then evaluated as iterated integrals.

To evaluate

$$\iiint\limits_D f(r, \theta, z)\, dV$$

over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

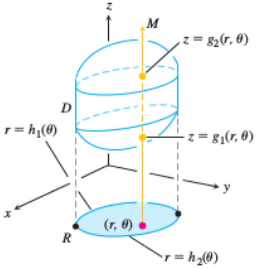
Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



How to Integrate in Cylindrical Coordinates : The z-Limits of Integration

Draw a line M through a typical point (r, θ) of R parallel to the z-axis.

As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limts of integration.

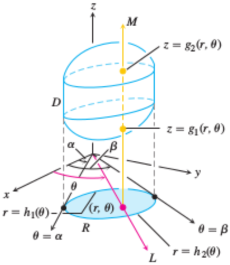


How to Integrate in Cylindrical Coordinates : The r-Limits of Integration

Draw a ray L through (r, θ) from the origin.

The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$.

These are the r-limits of integration.



How to Integrate in Cylindrical Coordinates : The θ-Limits of Integration

As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$.

These are the θ-limits of integration.

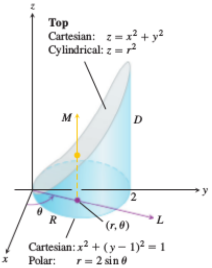
The integral is

$$\iiint\limits_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

Example 2.

Let $f(r, \theta, z)$ be a function defined over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y-1)^2 = 1$. Its polar coordinate equation is $r = 2\sin\theta$.



How to Integrate in Cylindrical Coordinates - An Example

We find the limits of integration, starting with the z-limits. A line M through typical point (r, θ) in R parallel to the z-axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next we find the r-limits of integration. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2 \sin \theta$.

Finally we find the θ-limits of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$.

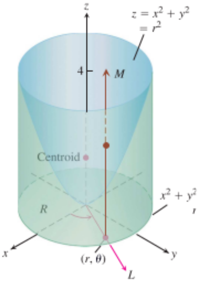
The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2 \sin \theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

Example 3.

Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the $xy-$plane.

Solution : We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$.



Solution (contd...)

Its base R is the disk $0 \leq r \leq 2$ in the $xy-$plane. The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M.

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

The z-limits. A line M through a typical point (r, θ) in the base parallel to the z-axis enters the solid at $z = 0$ and leaves at $z = r^2$.

The $r - limits$. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.

The $\theta - limits$. As L sweeps over the base like a clock hand, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$.

Solution (contd...)

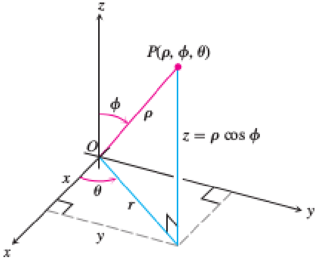
The value of M_{xy} is

$$\begin{aligned} M_{xy} &= \int_0^{2\pi}\int_0^2\int_0^{r^2} z\ dz\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^2 \frac{r^5}{2} dr\ d\theta = \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi}\frac{16}{3}d\theta = \frac{32\pi}{3}. \end{aligned}$$

The value of M is

$$\begin{aligned} M &= \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\ r\ dr\ d\theta = \int_0^{2\pi}\int_0^2 [z]_0^{r^2}\ r\ dr\ d\theta \\ &= \int_0^{2\pi}\int_0^2 r^3 dr\ d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4d\theta = 8\pi. \end{aligned}$$

Therefore $\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3}$, and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies outside the solid.



Spherical coordinates locate points in space with two angles and one distance.

The first coordinate, $\rho = |\overrightarrow{OP}|$, is the point's distance from the origin.

Unlike r, the variable ρ is never negative.

Spherical Coordinates

The scond coordinate, ϕ, is the angle $|\overrightarrow{OP}|$ makes with the positive z-axis. It is required to lie in the interval $[0, \pi]$.

The third coordinate is the angle θ as measured in cylindrical coordinates.

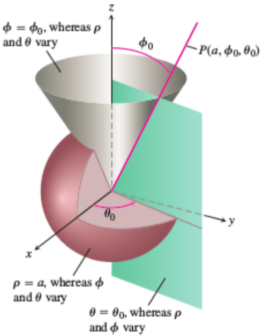
Definition 4.

Spherical Coordinates *represent a point P in space by ordered triples* (ρ, ϕ, θ) *in which*

1. ρ *is the distance from P to the origin.*
2. ϕ *is the angle* $\overrightarrow{OP}$ *makes with the positive z-axis* $(0 \leq \phi \leq \pi)$.
3. θ *is the angle from cylindrical coordinates.*

On maps of the Earch, θ is related to the meridian of a point on the Earth and ϕ to its latitute, while ρ is related to elevation above the Earth's surface.

The equation $\rho = a$ describes the sphere of radius a centered at the origin



The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis.

Here is an iterpretation to include the xy-plane as the cone $\phi = \pi/2$.

If ϕ_0 is greater than $\pi/2$, the cone $\phi = \phi_0$ opens downward.

The equation $\theta = \theta_0$ describes the half-plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates :

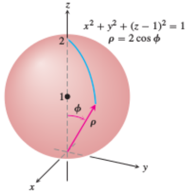
$$r = \rho \sin\phi, \quad x = r \cos\theta = \rho \sin\phi \cos\theta,$$

$$z = \rho \cos\phi, \quad y = r \sin\theta = \rho \sin\phi \sin\theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

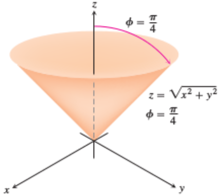
A spherical coordinate equation for the sphere $x^2 + (y-1)^2 + z^2 = 1$ is

$$\rho = 2 \sin\phi \sin\theta.$$



A spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$ is

$$\phi = \pi/4.$$



Spherical Coordinates

Spherical coordinates are good for describing spheres centered at the origin, half-planes hinged along the z-axis, and cones whose vertices lie at the origin and whose axes lie along the z-axis.

Surfaces like these have equations of constant coordinate value :

$\rho = 4$	Sphere, radius 4, center at origin
$\phi = \pi/3$	Cone opening up from the origin, making an angle of $\pi/3$ radians with the positive z-axis
$\theta = \pi/3$	Half-plane, hinged along the z-axis, making an angle of $\pi/3$ radians with the positive x-axis.

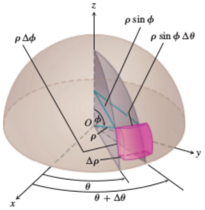
Spherical Coordinates

When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges.

The size of the kth spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given by changes by $\Delta\rho_k, \Delta\theta_k$, and $\Delta\phi_k$ in ρ, θ, and ϕ.

Such a spherical wedge has one edge a circular arc of length $\rho_k \Delta\phi_k$, another edge a circular arc of length $\rho_k \sin\phi_k \Delta\theta_k$, and thickness $\Delta\rho_k$.

The spherical wedge closed appropriates a cube of these dimensions when $\Delta\rho_k, \Delta\theta_k$, and $\Delta\phi_k$ are all small.



It can be shown that the volume of this spherical wedge is ΔV_k is

$$\Delta V_k = \rho_K^2 \sin \phi_k \Delta \rho_k \Delta \phi_k \Delta \theta_k$$

for $(\rho_k, \phi_k, \theta_k)$ a point chosen inside the wedge.

The corresponding Riemann sum for a function $F(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} F(\rho_k, \phi_k, \theta_k) \rho_k^2 \sin \phi_k \Delta \rho_k \Delta \phi_k \Delta \theta_k.$$

As the norm of a partition approaches zero, and the spherical wedges get smaller, the Riemann sums have a limit when F is continuous :

$$\lim_{n \to \infty} S_n = \iiint_D F(\rho, \phi, \theta)\, dV = \iiint_D F(\rho, \phi, \theta)\, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

Spherical Coordinates

In spherical coordinates, we have

$$dV = \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta.$$

To evaluate integrals in spherical coordinates, we usually integrate first with respect to ρ.

The procedure for finding the limits of integration is shown below.

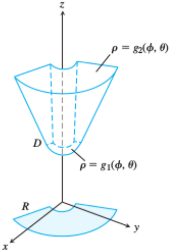
We restrict our attention to integrating over domains that are solids of revolution about the z-axis (or portions thereof) and for which the limits for θ and ϕ are constant.

To evaluate

$$\iiint\limits_{D} f(\rho, \phi, \theta)\; dV$$

over a region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ, take the following steps.

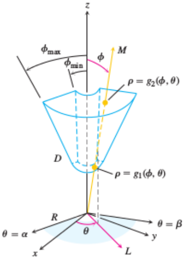
Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.



How to Integrate in Spherical Coordinates : ρ-Limits of Integration

Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L).

The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.



How to Integrate in Spherical Coordinates : ϕ and θ-Limits of Integration

ϕ-Limits of Integration

For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{min}$ to $\phi = \phi_{max}$. These are the ϕ-limits of integration.

θ-Limits of Integration

The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration.

The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

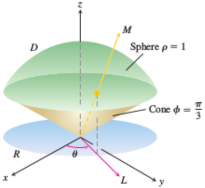
Example 5.

Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

The volume is

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta,$$

the integral $f(\rho, \phi, \theta) = 1$ over D. To find the limits of integration for evaluating the integral, we begin by sketching D and its projection R on the xy-plane.



How to Integrate in Spherical Coordinates - An Example

We draw a ray M from the origin through D making an angle ϕ with the positive z-axis.

We also draw L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive z-axis. For any given θ, the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$. The ray L sweeps over R as θ runs from 0 to 2π.

The volume is

$$V = \iiint_D \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta = \pi/3.$$

Example 6.

A solid of constant density $\delta = 1$ occupies the region D in Example 5. Find the solid's moment of intertia about the z-axis.

Solution : In rectangular coordinates, the moment is

$$I_z = \iiint (x^2 + y^2)\, dV.$$

In spherical coordinates,
$x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Hence,

$$I_z = \iiint \left(\rho^2 \sin^2\phi\right) \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta = \iiint \rho^4 \sin^3\phi \; d\rho \; d\phi \; d\theta.$$

Solution (contd...)

For the region in Example 5, this becomes

$$
\begin{aligned}
I_z &= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^5}{5}\right]_0^1 \sin^3\phi \, d\phi \, d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/3}\left(1-\cos^2\phi\right)\sin \ \phi \, d\phi \, d\theta \ = \frac{1}{5}\int_0^{2\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta \\
&= \frac{1}{5}\int_0^{2\pi}\left(-\frac{1}{2}+1+\frac{1}{24}-\frac{1}{3}\right)d\theta = \frac{1}{5}\int_0^{2\pi}\frac{5}{24}d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}.
\end{aligned}
$$

Exercise 7.

Evaluate the cylindrical coordinate integrals in the following exercises.

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\ r\ dr\ d\theta$
2. $\displaystyle\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\ dz\ r\ dr\ d\theta$
3. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{1/\sqrt{2-r^2}} 3\ dz\ r\ dr\ d\theta$
4. $\displaystyle\int_0^{2\pi}\int_0^1\int_{1/2}^{1/2} \left(r^2\ \sin^2\theta + z^2\right) dz\ r\ dr\ d\theta$

Solution for the Exercise 7

1. $\int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 [r(2-r^2)^{1/2} - r^2] dr\, d\theta =$
   $\int_0^{2\pi} \left[-\frac{1}{3}(2-r^2)^{3/2} - \frac{r^3}{3}\right]_0^1 d\theta = \int_0^{2\pi} \left(\frac{2^{3/2}}{3} - \frac{2}{3}\right) d\theta = \frac{4\pi(\sqrt{2}-1)}{3}$
2. $\int_0^{\pi} \int_0^{\theta/\pi} \int_{\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\, dz\, r\, dr\, d\theta = \int_0^{\pi} \int_0^{\theta/\pi} \frac{1}{2}\left[9(4-r^2) - (4-r^2)\right] r\, dr\, d\theta =$
   $4\int_0^{\pi} \int_0^{\theta/\pi} (4r - r^3)\, dr\, d\theta = 4\int_0^{\pi} \left[2r^2 - \frac{r^4}{4}\right]_0^{\theta/\pi} = 4\int_0^{\pi} \left(\frac{2\theta^2}{\pi^2} - \frac{\theta^4}{4\pi^4}\right) d\theta = \frac{37\pi}{15}$
3. $\int_0^{2\pi} \int_0^1 \int_r^{(2-r^2)^{-1/2}} 3\, dzr\, dr\, d\theta = 3\int_0^{2\pi} \int_0^1 \left[r(2-r^2)^{-1/2} - r^2\right] dr\, d\theta =$
   $3\int_0^{2\pi} \left[-(2-r^2)^{1/2} - \frac{r^3}{3}\right]_0^1 d\theta = 3\int_0^{2\pi} \left(\sqrt{2} - \frac{4}{3}\right) d\theta = \pi(6\sqrt{2} - 8)$
4. $\int_0^{2\pi} \int_0^1 \int_{-1/2}^{1/2} (r^2 \sin^2\theta + z^2) dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^1 (r^3 \sin^2\theta + \frac{r}{12}) dr\, d\theta =$
   $\int_0^{2\pi} \left(\frac{\sin^2\theta}{4} + \frac{1}{24}\right) d\theta = \frac{\pi}{3}$

Exercise 8.

The integrals we have seen so far suggest that there are preferred orders of integration for cylindrical coordinates, but other orders usually work well and are occasionally easier to evaluate. Evaluate the integrals in the following exercises.

1. $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\, dr\, dz\, d\theta$
2. $\displaystyle\int_0^2\int_{r-2}^{\sqrt{4-r^2}} (r\ \sin\theta + 1)\, r\, d\theta\, dz\, dr$
3. *Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.*
   (a) $dz\ dr\ d\theta$ (b) $dr\ dz\ d\theta$ (c) $d\theta\ dz\ dr$
4. *Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.*
   (a) $dz\ dr\ d\theta$ (b) $dr\ dz\ d\theta$ (c) $d\theta\ dz\ dr$

Solution for (1.) and (2.) in Exercise 8

1. $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\,dr\,dz\,d\theta = \int_0^{2\pi}\int_0^3 \frac{z^4}{324}\,dz\,d\theta = \int_0^{2\pi}\frac{3}{20}\,d\theta = \frac{3\pi}{10}$

2. $\displaystyle\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2x}(r\sin\theta+1)r\,d\theta\,dz\,dr = \int_0^2\int_{r-2}^{\sqrt{4-r^2}} 2\pi r\,dz\,dr =$

   $\displaystyle 2\pi\int_0^2\left[r(4-r^2)^{1/2}-r^2+2r\right]\,dr = 2\pi\left[-\frac{1}{3}(4-r^2)^{3/2}-\frac{r^3}{3}+r^2\right]_0^2 =$

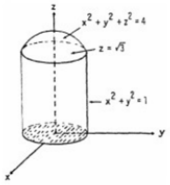
   $\displaystyle 2\pi\left[-\frac{8}{3}+4+\frac{1}{3}(4)^{3/2}\right] = 8\pi$

Solution for (3.) in Exercise 8

(a) $\displaystyle\int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{4-r^2}} dz\, r\, dr\, d\theta$

(b) $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{1} r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_{\sqrt{3}}^{2}\int_0^{\sqrt{4-r^2}} r\, dr\, dz\, d\theta$

(c) $\displaystyle\int_0^{1}\int_0^{\sqrt{4-r^2}}\int_0^{2\pi} r\, d\theta\, dz\, dr$

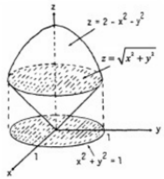


Solution for (4.) in Exercise 8

(a) $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{2-r^2} dz\, r\, dr\, d\theta$

(b) $\displaystyle\int_0^{2\pi}\int_0^1\int_0^z r\, dr\, dz\, d\theta + \int_0^{2\pi}\int_1^2\int_0^{\sqrt{2-z}} r\, dr\, dz\, d\theta$

(c) $\displaystyle\int_0^1\int_r^{2-r^2}\int_0^{2\pi} r\, d\theta\, dz\, dr$



Exercise 9.

1. *Give the limits of integration for evaluating the integral*

$$\iiint f(r,\theta,z)\,dz\ r\ dr\ d\theta$$

as an iterated integral over the region that is bounded below by the plane $z = 0$, *on the side by the cylinder* $r = \cos\theta$, *and on top by the paraboloid* $z = 3r^2$.

2. *Convert the integral*

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}\left(x^2+y^2\right)dz\ dx\ dy$$

to an equivalent integral in cylindrical coordinates and evaluate the result.

Solution for the Exercise 9

1. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{3r^2} f(r,\theta,z)\,dz\,r\,dr\,d\theta$
2.

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2}\int_0^{1}\int_0^{r\cos\theta} r^3\,dz\,dr\,d\theta &= \int_{-\pi/2}^{\pi/2}\int_0^{1} r^4\cos\theta\,dr\,d\theta \\
&= \frac{1}{5}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta \\
&= \frac{2}{5}
\end{aligned}$$

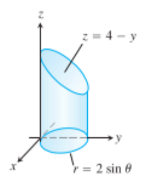
Exercise 10.

In the following exercises, set up the iterated integral for evaluating

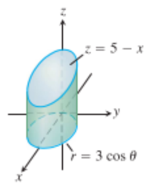
$$\iiint_D f(r, \theta, z)\, dz\ r\ dr\ d\theta$$

over the given region D.

1. *D is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the $xy-$ plane and whose top lies in the plane $z = 4 - y$.*



2. *D is the right circular cylinder whose base is the circle $r = 3\cos\theta$ and whose top lies in the plane $z = 5 - x$.*



Solution for the Exercise 10

1. $$\int_0^{\pi} \int_0^{2\sin\theta} \int_0^{4+r\sin\theta} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$
2. $$\int_{-\pi/2}^{\pi/2} \int_0^{3\cos\theta} \int_0^{5r\cos\theta} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

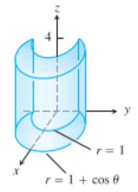
Exercise 11.

In the following exercises, set up the iterated integral for evaluating

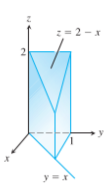
$$\iiint_D f(r, \theta, z)\, dz\; r\; dr\; d\theta$$

over the given region D.

1. *D is the solid right cylinder whose base is the region in the $xy-$ plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.*



2. *D is the prism whose base is the triangle in the $xy-$plane bounded by the $y-$axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$.*



Solution for the Exercise 11

1. $$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} \int_{0}^{4} f(r, \theta, z)\, dz\, r\, dr\, d\theta$$
2. $$\int_{0}^{\pi/4} \int_{0}^{\sec\theta} \int_{0}^{2+r\sin\theta} f(r, \theta, z)\, dz\, r\, dr\, d\theta$$

Exercise 12.

Evaluate the spherical coordinate integrals in the following exercises.

1. $$\int_0^{\pi} \int_0^{\pi} \int_0^{2\sin\phi} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$
2. $$\int_0^{2\pi} \int_0^{\pi} \int_0^{(1-\cos\phi)/2} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$
3. $$\int_0^{3\pi/2} \int_0^{\pi} \int_0^{1} 5\rho^3 \sin^3\phi \; d\rho \; d\phi \; d\theta$$
4. $$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} (\rho\cos\phi)\, \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

1. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{8}{3}\int_0^{\pi}\int_0^{\pi}\sin^4\phi\,d\phi\,d\theta =$
$\displaystyle\frac{8}{3}\int_0^{\pi}\left(\left[-\frac{\sin^3\phi\cos\phi}{4}\right]_0^{\pi}+\frac{3}{4}\int_0^{\pi}\sin^2\phi\,d\phi\right)d\theta = 2\int_0^{\pi}\int_0^{\pi}\sin^2\phi\,d\phi\,d\theta =$
$\displaystyle\int_0^{\pi}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi}d\theta = \int_0^{\pi}\pi d\theta = \pi^2$

2. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-cos\phi)/2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{24}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi d\,\phi\,d\theta =$
$\displaystyle\frac{1}{96}\int_0^{2\pi}\left[(1-\cos\phi)^4\right]_0^{\pi}\,d\theta = \frac{1}{96}\int_0^{2\pi}(2^4-0)\,d\theta = \frac{16}{96}\int_0^{2\pi}d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}$

3. $\displaystyle\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1}5\rho^3\sin^3\phi\,d\rho\,d\phi\,d\theta = \frac{5}{4}\int_0^{3\pi/2}\int_0^{\pi}\sin^3\phi\,d\phi\,d\theta =$
$\displaystyle\frac{5}{4}\int_0^{3\pi/2}\left(\left[-\frac{\sin^2\phi\cos\phi}{3}\right]_0^{\pi}+\frac{2}{3}\int_0^{\pi}\sin\phi\,d\phi\right)d\theta = \frac{5}{6}\int_0^{3\pi/2}[-\cos\phi]_0^{\pi}d\theta =$
$\displaystyle\frac{5}{3}\int_0^{3\pi/2}d\theta = \frac{5\pi}{2}$

4. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi}\rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/4}\tan\phi\sec^2\phi\,d\phi\,d\theta =$
$\displaystyle\frac{1}{4}\int_0^{2\pi}[\frac{1}{2}\tan^2\phi]_0^{\pi/4}\,d\theta = \frac{1}{8}\int_0^{2\pi}d\theta = \frac{\pi}{4}$

Exercise 13.

The previous integrals suggest there are preferred orders of integration for spherical coordinates, but other orders give the same value and are occasionally easier to evaluate. Evaluate the integrals in the following exercises.

1. $\displaystyle\int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \; d\phi \; d\theta \; d\rho$
2. $\displaystyle\int_0^1 \int_0^{\pi} \int_0^{\pi/4} 12\rho \sin^3 \phi \; d\phi \; d\theta \; d\rho$

Solution for the Exercise 13

1. $\int_0^2 \int_{-\pi}^0 \int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho = \int_0^2 \int_{-\pi}^0 \rho^3 [-\frac{\cos 2\phi}{2}]_{\pi/4}^{\pi/2} d\theta \, d\rho = \int_0^2 \int_{-\pi}^0 \frac{\rho^3}{2} \, d\theta \, d\rho =$
$\int_0^2 \frac{\rho^3 \pi}{2} \, d\rho = \left[\frac{\pi \rho^3}{8} \right]_0^2 = 2\pi$

2. $\int_0^1 \int_0^\pi \int_0^{\pi/4} 12\rho \sin^3 \phi \, d\phi \, d\theta \, d\rho =$
$\int_0^1 \int_0^\pi \left(12\rho \left[\frac{-\sin^2 \rho \cos \phi}{3} \right]_0^{\pi/4} + 8\rho \int_0^{\pi/4} \sin \phi \, d\phi \right) d\theta \, d\rho =$
$\int_0^1 \int_0^\pi \left(-\frac{2\rho}{\sqrt{2}} - 8\rho [\cos \phi]_0^{\pi/4} \right) d\theta \, d\rho = \int_0^1 \int_0^\pi \left(8\rho - \frac{10\rho}{\sqrt{2}} \right) d\theta \, d\rho =$
$\pi \int_0^1 \left(8\rho - \frac{10\rho}{\sqrt{2}} \right) d\rho = \pi \left[4\rho^2 - \frac{5\rho^2}{\sqrt{2}} \right]_0^1 = \frac{\left(4\sqrt{2} - 5 \right) \pi}{\sqrt{2}}$

Exercise 14.

1. *Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.*

   (a) $d\rho\ d\phi\ d\theta$ (b) $d\phi\ d\rho\ d\theta$

2. *Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.*

   (a) $d\rho\ d\phi\ d\theta$ (b) $d\phi\ d\rho\ d\theta$

Solution for (1.) in Exercise 14

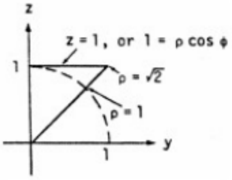
(a) $x^2 + y^2 = 1 \Rightarrow \rho^2 \sin^2 \phi = 1$, and $\rho \sin \phi = 1 \Rightarrow \rho = \csc \phi$; thus

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^{2} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta + \int_0^{2\pi} \int_{\pi/6}^{\pi/2} \int_0^{\csc \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

(b) $\displaystyle\int_0^{2\pi} \int_1^{2} \int_{\pi/6}^{\sin^{-1}(1/\rho)} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi} \int_0^{2} \int_0^{\pi/6} \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta$

(a) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(b) $\int_0^{2\pi}\int_0^{1}\int_0^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta + \int_0^{2\pi}\int_1^{\sqrt{2}}\int_{\cos^{-1}(1/\rho)}^{\pi/4} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$



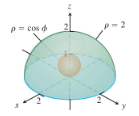
Exercise 15.

In the following exercises,

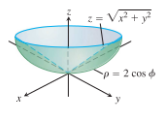
(a) *find the spherical coordinate limits for the integral that calculates the volume of the given solid and then*

(b) *evaluate the integral.*

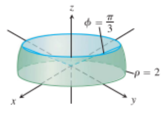
1. *The solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, z \geq 0$.*



2. *The solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$.*



3. *The solid bounded below by the $xy-$plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$.*



Solution for the Exercise 15

1. $V = \int_0^{2\pi} \int_0^{\pi/2} \int_{\cos\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \dfrac{1}{3} \int_0^{2\pi} \int_0^{\pi/2} (8 - \cos^3\phi) \sin\phi \, d\phi \, d\theta =$
   $\dfrac{1}{3} \int_0^{2\pi} \left[-8\cos\phi + \dfrac{\cos^4\phi}{4} \right]_0^{\pi/2} d\theta = \dfrac{1}{3} \int_0^{2\pi} \left(8 - \dfrac{1}{4} \right) d\theta = (\dfrac{31}{12})(2\pi) = \dfrac{31\pi}{6}$
2. $V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \dfrac{8}{3} \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \cos^3\phi \sin\phi \, d\phi \, d\theta =$
   $\dfrac{8}{3} \int_0^{2\pi} \left[-\dfrac{\cos^4\phi}{4} \right]_{\pi/4}^{\pi/2} d\theta = \left(\dfrac{8}{3}\right) \left(\dfrac{1}{16}\right) \int_0^{2\pi} d\theta = \dfrac{1}{6}(2\pi) = \dfrac{\pi}{3}$
3. $V = \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \dfrac{8}{3} \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \sin\phi \, d\phi \, d\theta =$
   $\dfrac{8}{3} \int_0^{2\pi} [-\cos\phi]_{\pi/3}^{\pi/2} \, d\theta = \dfrac{4}{3} \int_0^{2\pi} d\theta = \dfrac{8\pi}{3}$

Exercise 16.

1. *Set up triple integrals for the volume of the sphere* $\rho = 2$ *in*
   (a) *spherical,*
   (b) *cylindrical, and*
   (c) *rectangular coordinates.*
2. *Let D be the smaller cap cut from a solid ball of radius 2 units by a plane* 1 *unit from the center of the sphere. Express the volume of D as an iterated triple integral in*
   (a) *spherical,*
   (b) *cylindrical, and*
   (c) *rectangular coordinates. Then*
   (d) *find the volume by evaluating one of the three triple integrals.*

Solution for (1.) in Exercise 16

(a) $8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(b) $8\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$

(c) $8\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$

(a) $V = \displaystyle\int_0^{2\pi} \int_0^{\pi/3} \int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

(b) $V = \displaystyle\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta$

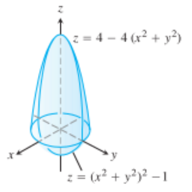
(c) $V = \displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} dz \, dy \, dx$

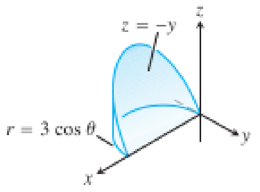
(d) $V = \displaystyle\int_0^{2\pi} \int_0^{\sqrt{3}} \left[r(4-r^2)^{1/2} - r \right] dr \, d\theta = \int_0^{2\pi} \left[-\frac{(4-r^2)^{3/2}}{3} - \frac{r^2}{2} \right]_0^{\sqrt{3}} d\theta =$

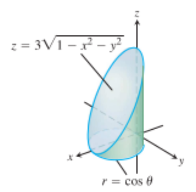
$\displaystyle\int_0^{2\pi} \left(-\frac{1}{3} - \frac{3}{2} + \frac{4^{1/2}}{3} \right) d\theta = \frac{5}{6} \int_0^{2\pi} d\theta = \frac{5\pi}{3}$

Exercise 17.

Find the volumes of the solids in the following exercises.

1. 

2. 

3. 

Solution for the Exercise 17

1. $V = 4\int_0^{\pi/2}\int_0^1\int_{r^4-1}^{4-4r^2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 (5r - 4r^3 - r^5) dr\, d\theta =$
$4\int_0^{\pi/2}(\frac{5}{2} - 1 - \frac{1}{6}) d\theta = 4\int_0^{\pi/2} d\theta = \frac{8\pi}{3}$
2. $V = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta}\int_0^{-r\sin\theta} dz\, r\, dr\, d\theta = \int_{3\pi/2}^{2\pi}\int_0^{3\cos\theta} -r^2 \sin\theta\, dr\, d\theta =$
$\int_{3\pi/2}^{2\pi}(-9\cos^3\theta)(\sin\theta) d\theta = \left[\frac{9}{4}\cos^4\theta\right]_{3\pi/2}^{2\pi} = \frac{9}{4} - 0 = \frac{9}{4}$
3. $V = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta}\int_0^r dz\, r\, dr\, d\theta = 2\int_{\pi/2}^{\pi}\int_0^{-3\cos\theta} r^2\, dr\, d\theta = \frac{2}{3}\int_{\pi/2}^{\pi} -27\cos^3\theta\, d\theta =$
$-18\left(\left[\frac{\cos^2\theta\sin\theta}{3}\right]_{\pi/2}^{\pi} + \frac{2}{3}\int_{\pi/2}^{\pi}\cos\theta d\theta\right) = -12[\sin\theta]_{\pi/2}^{\pi} = 12$

Exercise 18.

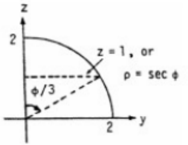
1. Sphere and cones : *Find the volume of the portion of the solid sphere* $\rho \leq a$ *that lies between the cones* $\phi = \pi/3$ *and* $\phi = 2\pi/3$.
2. Cylinder and paraboloid : *Find the volume of the region bounded below by the plane* $z = 0$, *laterally by the cylinder* $x^2 + y^2 = 1$, *and above by the paraboloid* $z = x^2 + y^2$.
3. Cylinder and paraboloids : *Find the volume of the region bounded below by the paraboloid* $z = x^2 + y^2$, *laterally by the cylinder* $x^2 + y^2 = 1$, *and above by the paraboloid* $z = x^2 + y^2 + 1$.
4. Sphere and cylinder : *Find the volume of the region that lies inside the sphere* $x^2 + y^2 + z^2 = 2$ *and outside the cylinder* $x^2 + y^2 = 1$.
5. Sphere and plane : *Find the volume of the smaller region cut from the solid sphere* $\rho \leq 2$ *by the plane* $z = 1$.

Solution for (1), (2), (3) and (4) in Exercise 18

1. $V = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \int_0^a \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \frac{a^3}{3} \sin\phi \, d\phi \, d\theta =$
$\frac{a^3}{3} \int_0^{2\pi} [-\cos\phi]_{\pi/3}^{2\pi/3} d\theta = \frac{a^3}{3} \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\right) d\theta = \frac{2\pi a^3}{3}$
2. $V = 4 \int_0^{\pi/2} \int_0^1 \int_0^{r^2} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r^3 \, dr \, d\theta = \int_0^{\pi/2} d\theta = \frac{\pi}{2}$
3. $V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{r^2+1} dz \, r \, dr \, d\theta = 4 \int_0^{\pi/2} \int_0^1 r \, dr \, d\theta = 2 \int_0^{\pi/2} d\theta = \pi$
4. $V = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^r dz \, r \, dr \, d\theta = 8 \int_0^{\pi/2} \int_1^{\sqrt{2}} r^2 \, dr \, d\theta = 8 \left(\frac{2\sqrt{2}-1}{3}\right) \int_0^{\pi/2} d\theta =$
$\frac{4\pi(2\sqrt{2}-1)}{3}$

Solution for (5) in Exercise 18

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/3} (8\sin\phi - \tan\phi\sec^2\phi)\, d\phi\, d\theta \\
&= \frac{1}{3}\int_0^{2\pi} [-8\cos\phi - \frac{1}{2}\tan^2\phi]_0^{\pi/3} d\theta \\
&= \frac{1}{3}\int_0^{2\pi} [-4 - \frac{1}{2}(3) + 8] d\theta \\
&= \frac{1}{3}\int_0^{2\pi} \frac{5}{2}\, d\theta \\
&= \frac{5}{6}(2\pi) = \frac{5\pi}{3}
\end{aligned}
$$



Exercise 19.

1. Cylinder and planes : *Find the volume of the region enclosed by the cylinder* $x^2 + y^2 = 4$ *and the planes* $z = 0$ *and* $x + y + z = 4$.
2. Region trapped by paraboloids : *Find the volume of the region bounded above by the paraboloid* $z = 5 - x^2 - y^2$ *and below by the paraboloid* $z = 4x^2 + 4y^2$.
3. Paraboloid and cylinder : *Find the volume of the region bounded above by the paraboloid* $z = 9 - x^2 - y^2$, *below by the* $xy-$ *plane, and lying outside the cylinder* $x^2 + y^2 = 1$.
4. Sphere and paraboloid : *Find the volume of the region bounded above by the sphere* $x^2 + y^2 + z^2 = 2$ *and below by the paraboloid* $z = x^2 + y^2$.

Solution for the Exercise 19

1. $V = \int_0^{2\pi} \int_0^2 \int_0^{4-r\cos\theta - r\sin\theta} dz\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^2 [4r - r^2(\cos\theta + \sin\theta)] dr\, d\theta = \frac{8}{3} \int_0^{2\pi} (3 - \cos\theta - \sin\theta) d\theta = 16\pi$
2. The paraboloids intersect when $4x^2 + 4y^2 = 5 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$ and $z = 4$ $\Rightarrow V = 4 \int_0^{\pi/2} \int_0^1 \int_{4r^2}^{5-r^2} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^1 (5r - 5r^3)\, dr\, d\theta = 20 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta = 5 \int_0^{\pi/2} d\theta = \frac{5\pi}{2}$
3. The paraboloid intersects the xy-plane when $9 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 9 \Rightarrow V = 4 \int_0^{\pi/2} \int_1^3 \int_0^{9-r^2} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_1^3 (9r - r^3) dr\, d\theta = 4 \int_0^{\pi/2} \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_1^3 d\theta = 4 \int_0^{\pi/2} (\frac{81}{4} - \frac{17}{4}) d\theta = 64 \int_0^{\pi/2} d\theta = 32\pi$
4. The sphere and paraboloid intersect when $x^2 + y^2 + z^2 = 2$ and $z = x^2 + y^2 \Rightarrow z^2 + z - 2 = 0 \Rightarrow (z+2)(z-1) = 0 \Rightarrow z = 1$ or $z = -2 \Rightarrow z = 1$ since $z \geq 0$. Thus, $x^2 + y^2 = 1$ and the volume is given by the triple integral $V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} dz\, r\, dr\, d\theta = 4 \int_0^{\pi/2} \int_0^1 \left[r(2 - r^2)^{1/2} - r^3\right] dr\, d\theta = 4 \int_0^{\pi/2} \left[-\frac{1}{3}(2 - r^2)^{3/2} - \frac{r^4}{4}\right]_0^1 d\theta = 4 \int_0^{\pi/2} \left(\frac{2\sqrt{2}}{3} - \frac{7}{12}\right) d\theta = \frac{\pi\left(8\sqrt{2}-7\right)}{6}$

Exercise 20.

1. *Find the average value of the function* $f(r, \theta, z) = r$ *over the solid ball bounded by the sphere* $r^2 + z^2 = 1$*. (This is the sphere* $x^2 + y^2 + z^2 = 1$*.)*
2. *Find the average value of the function* $f(\rho, \phi, \theta) = \rho \cos \phi$ *over the solid solid upper ball* $\rho \leq 1, 0 \leq \phi \leq \pi/2$*.*

Solution for the Exercise 20

1.

$$\begin{aligned}
average &= \frac{1}{\left(\frac{4\pi}{3}\right)} \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^2 \, dz \, dr \, d\theta \\
&= \frac{3}{4\pi} \int_0^{2\pi} \int_0^1 2r^2 \sqrt{1-r^2} dr \, d\theta \\
&= \frac{3}{2\pi} \int_0^{2\pi} \left[\frac{1}{8} \sin^{-1} r - \frac{1}{8} r \sqrt{1-r^2}(1-2r^2) \right]_0^1 d\theta = \frac{3}{16\pi} \int_0^{2\pi} (\frac{\pi}{2} + 0) d\theta \\
&= \frac{3}{32} \int_0^{2\pi} d\theta = \left(\frac{3}{32}\right)(2\pi) = \frac{3\pi}{16}
\end{aligned}$$

2.

$$\begin{aligned}
average &= \frac{1}{\left(\frac{2\pi}{3}\right)} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \frac{3}{8\pi} \int_0^{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi \, d\phi \, d\theta \\
&= \frac{3}{8\pi} \int_0^{2\pi} \left[\frac{\sin^2\phi}{2} \right]_0^{\pi/2} d\theta = \frac{3}{16\pi} \int_0^{2\pi} d\theta = \left(\frac{3}{16\pi}\right)(2\pi) = \frac{3}{8}
\end{aligned}$$

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